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Fluid oscillations in an open, flexible container

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Abstract. In this paper we study normal oscillation modes of an incompressible fluid in an open container, part of the wall of which may be flexible. The flexible part of the container wall is modelled by a membrane. We first investigate the eigenfrequencies of an inviscid fluid in a flexible container. We are able to show, by analytical means, that the eigenfrequencies of an inviscid fluid decrease when part of the rigid container wall is replaced by a membrane. The problem of viscous fluid oscillations in a flexible container is then studied numerically using the finite-element technique. Two different types of eigenmodes are observed: free-surface oscillation modes and structural vibration modes. The dependence of the two modes on the Bond number (measure of the ratio of gravitational and tension forces) and the Reynolds number is investigated.

1. Introduction

Fluids oscillating in an open, flexible container occur in a wide variety of situations ranging from propellant motion in tanks of air- and spacecraft to oil and water oscillations in storage tanks. The flexible container walls may be deflected under the action of fluid forces exerted on the walls. Since the motion of the fluid is in turn affected by the moving container wall, this type of problem is generally known as a fluid-structure interaction problem.

The first difficulty one encounters when dealing with the fluid-structure interaction problem, is how to model the system of fluid and structure. Depending on the characteristics of the structure it may be modelled by for example a membrane, a shell, a thick plate or a solid obeying the equations of elasticity. Likewise, depending on the type of fluid interacting with the structure, viscous effects may either be neglected or may have to be taken into account and the incompressibility constraint may or may not have to be imposed. It will be clear that any combination of the structure and fluid models will lead to problems with different characteristics and solutions with different properties. This leads to the second major difficulty, namely how to solve the, often non-linear, partial differential equations. Again depending on the system to be investigated, one may either have to solve the complete non-linear set of equations or linearize the equations and solve the resulting linear problem. Regarding the modelling problem, one can find virtually every structure model in the literature while the fluid part of the model is often dealt with by neglecting viscous effects. Concerning the solution of the equations, the emphasis has been on solving the linear problem due to the inherent difficulties of solving non-linear partial differential equations and modelling non-linear interactions. Lately however, due to the increase in computational power and the development of new numerical techniques, the non-linear problem has attracted much attention. In this paper we shall restrict ourselves to the linear problem. An overview of the literature concerning both the modelling and solution techniques regarding fluid-structure interaction problems, is given by Belytschko [1] and Zienkiewicz and Bettess [2].

When dealing with the linear fluid-structure interaction problem one is often interested in the eigenfrequencies of the coupled system; how does the fluid affect the eigenmodes of the structure and *vice versa*, and indeed how is the structure deflected when the fluid oscillates in one of its eigenmodes. Also of interest is what happens when one of the principal eigenmodes of the fluid is near a principal mode of the structure. Miles [3] was one of the first to deal with the eigenfrequencies of the fluid-structure problem. He investigated, using analytical techniques, how the eigenfrequencies of a cylindrical flexible container are affected when it contains an inviscid fluid. Coale [4] dealt with a similar problem of inviscid fluid oscillations in a cylindrical, elastic, massless shell, with particular emphasis on the modal displacements of the shell. Boujot [5, 6], Valid and Ohayon [7] and Berger et al. [8], among others, have investigated the problem of incompressible, inviscid fluid oscillations in a thick deformable shell, the motion of which is governed by the equations of linear elasticity. A number of existence and uniqueness results are obtained and various aspects of the numerical solution of the equations using the finite-element technique are investigated. Inviscid, compressible fluid oscillations wholly contained in a (partly) flexible container have been studied by Hamdi et al. [9], Morand and Ohayon [10], Geradin et al. [11] and Deneuvy [12].

In this paper we will consider the motion of an incompressibile, viscous fluid in an open flexible container. The container wall is modelled by a membrane following Schulkes and Cuvelier [13]. In Section 2 the problem to be considered is introduced and the governing equations are presented. Section 3 deals with the deflection of the membrane due to the hydrostatic pressure. In Section 4 we consider the related problem of inviscid fluid oscillations and are able to show analytically, that under certain assumptions the eigenfrequencies decrease when a rigid wall is replaced by a membrane. In Section 5 a number of qualitative properties of the spectral problem, which results from the linearized Navier–Stokes equations, are derived. We find that two types of normal modes exist, namely modes related to free-surface oscillations and modes related to membrane vibrations. The two types of modes have different damping characteristics and depend differently on the parameters. Section 6 deals with the discrete eigenvalue problem obtained after a finite-element discretization has been applied. Some general results concerning eigenvalues of the discrete eigenvalue problem are obtained. The eigenvalue problem is solved using an inverse iteration procedure. In Section 7 numerical results are presented. Particular emphasis is placed on the fluid motion resulting from normal mode oscillations and the damping characteristics of various types of normal modes. Both of these aspects of fluid-structure interaction problems have not received a great deal of attention so far. We also investigate the case in which the eigenfrequencies of the free-surface oscillations and membrane vibrations are close. The Reynolds number is found to be an important parameter in this case.

2. Problem formulation

Consider an open $2D$ container C with boundary denoted by ∂C. Let a part of the container be filled with a Newtonian, incompressible fluid of density ρ_f and kinematic viscosity ν. The region occupied by the fluid is denoted by Ω with boundary $\partial\Omega$. The free surface of the fluid is denoted by S and the area of fluid in contact with the container wall by $\Gamma = \partial\Omega \cap \partial C$. Some parts of the container wall may be flexible; let F and R denote the flexible and rigid parts respectively of the container wall in contact with the fluid, viz. $F \cup R = \Gamma$ (cf. Fig. 1). The equations describing the motion of the fluid in Ω are the

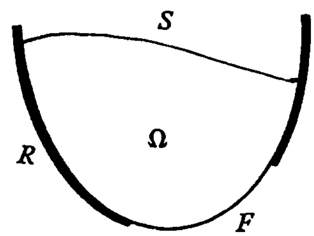

Fig. 1.

Navier–Stokes equations

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} + \frac{1}{\rho_f}\nabla p = \nu\nabla^2\mathbf{u} + \mathbf{f}\,, \tag{2.1}$$

and the incompressibility condition

$$\nabla\cdot\mathbf{u} = 0\,. \tag{2.2}$$

In the above equations $\mathbf{u}$ denotes the fluid velocity, p the pressure and $\mathbf{f}$ the external body force. Equations (2.1) and (2.2) have to be supplemented with a set of boundary conditions in order to be able to solve them. On the free surface S we prescribe normal and tangential stresses, viz.

$$\sigma_n = -p_g\,, \qquad \sigma_\tau = 0\,. \tag{2.3}$$

Here p_g denotes the outside gas pressure and σ_n, σ_τ denote the normal and tangential stresses respectively, given by

$$\sigma_n = (\boldsymbol{\sigma}\cdot\mathbf{n})\cdot\mathbf{n}\,, \qquad \sigma_\tau = (\boldsymbol{\sigma}\cdot\mathbf{n})\cdot\boldsymbol{\tau}\,,$$

in which $\mathbf{n}$ and $\boldsymbol{\tau}$ denote unit normal and tangential vectors respectively and $\boldsymbol{\sigma}$ is the Cauchy stress tensor (cf. Batchelor [14]). On the free surface we also have the kinematic condition

$$\frac{D}{Dt}\,\alpha(\mathbf{x}, t) = 0\,, \tag{2.4}$$

in which $\alpha(\mathbf{x}, t) = \mathbf{n}\cdot(\mathbf{x}_S - \mathbf{x}) = 0$ is the equation of the free surface S ($\mathbf{x}_S$ is the position vector of S) and D/Dt denotes the Lagrangian derivative.

On the rigid part of the container wall R the no-slip condition characteristic of viscous flows is prescribed, viz.

$$\mathbf{u} = 0\,. \tag{2.5}$$

The flexible part of the container wall is modelled by a membrane of density ρ_m and tension T. The boundary conditions on F become

$$\begin{aligned} \sigma_n &= -p_g + \frac{T}{R_c} - \rho_m\frac{\partial^2\beta}{\partial t^2}\,, \\ \mathbf{u}\cdot\boldsymbol{\tau} &= 0\,, \end{aligned} \tag{2.6}$$

in which R_c denotes the radius of curvature of the membrane considered positive when the corresponding centre of curvature lies outside the liquid. The equation of the membrane is given by $\beta(\mathbf{x}, t) = \mathbf{n}\cdot(\mathbf{x}_F - \mathbf{x}) = 0$ in which $\mathbf{x}_F$ denotes the position vector of F relative to some origin. On F we also have the kinematic condition

$$\frac{D}{Dt}\,\beta(\mathbf{x}, t) = 0\,. \tag{2.7}$$

The end points of the membrane are assumed fixed to rigid parts of the container wall so that the points at which F and R intersect the condition $\mathbf{u} = 0$ is prescribed. The last condition to be satisfied is the volume constraint – the quantity of liquid in the container, V_f, remains constant so that

$$\int_\Omega \mathrm{d}x = V_f \,. \tag{2.8}$$

Equations (2.1) and (2.2) together with boundary conditions (2.3)–(2.8) completely describe the motion of a viscous, incompressible fluid with a free boundary in a container of which part of the wall may be flexible. Although (2.1) and (2.2) can in principle be solved subject to (2.3)–(2.8), the solution of the complete set of non-linear equations is beyond the scope of this paper. We linearize the governing equations by considering only small perturbations from the steady state.

Consider the steady state $\mathbf{u} = 0$. The momentum equation reduces to

$$\frac{1}{\rho_f} \nabla p_0 = \mathbf{f} \quad \text{in } \Omega_0 \,, \tag{2.9}$$

the free surface condition becomes

$$-p_0 = -p_g \quad \text{on } S_0 \,, \tag{2.10}$$

and the condition on the flexible part of the container wall reads

$$-p_0 = -p_g + \frac{T}{R_{C_0}} \quad \text{on } F_0 \,. \tag{2.11}$$

In the above and subsequent equations the subscript 0 refers to steady state parameters. Equations (2.9)–(2.11) together with the volume condition (2.8) define the position of the steady state free surface S_0 and the shape of the flexible container wall F_0. We will take $\mathbf{f} = -g\hat{\mathbf{y}}$ where $\hat{\mathbf{y}}$ is the unit vector pointing along the positive y-axis and g the gravitational acceleration. Note that since we have neglected surface tension effects S_0 will be a flat surface perpendicular to the y-axis. Let us now perturb the steady state such that the position vectors of S and F are given by

$$\mathbf{x}_S = \mathbf{x}_{S_0} + \eta \mathbf{n}_S \,,$$

and

$$\mathbf{x}_F = \mathbf{x}_{F_0} + \xi \mathbf{n}_F \,.$$

The vectors $\mathbf{n}_S$ and $\mathbf{n}_F$ are the outward unit normals to S_0 and F_0 respectively and η and ξ denote small normal deviations from the steady state surfaces S_0 and F_0. The linearized equations in dimensionless form describing the fluid motion are then

$$\left.\begin{aligned} &\frac{\partial \mathbf{u}}{\partial t} + \nabla p = \frac{1}{Re} \nabla^2 \mathbf{u} \\ &\nabla \cdot \mathbf{u} = 0 \end{aligned}\right\} \quad \text{in } \Omega_0 \,. \tag{2.12}$$

and the linearized dimensionless boundary conditions become

$$\left.\begin{aligned} \sigma_n &= -\eta \\ \sigma_\tau &= 0 \\ \mathbf{u}\cdot\mathbf{n} &= \frac{\partial \eta}{\partial t} \end{aligned}\right\} \quad \text{on } S_0\,, \tag{2.13}$$

$$\mathbf{u} = 0 \qquad \text{on } R\,, \tag{2.14}$$

$$\left.\begin{aligned} \sigma_n &= -\xi\hat{\mathbf{y}}\cdot\mathbf{n}_F + \frac{1}{Bo}\left(\frac{\xi}{R_{C_0}^2} + \frac{\partial^2\xi}{\partial s^2}\right) - r_\rho\,\frac{\partial^2\xi}{\partial t^2} \\ \mathbf{u}\cdot\boldsymbol{\tau} &= 0 \\ \mathbf{u}\cdot\mathbf{n} &= \frac{\partial \xi}{\partial t} \end{aligned}\right\} \quad \text{on } F_0\,. \tag{2.15}$$

In the above equations $Re = lU/\nu$ is the Reynolds number (l is a length scale and $U = \sqrt{gl}$ a velocity scale), $Bo = \rho_f g l^2/T$ is the Bond number which is a measure of the ratio of gravitational and tension forces in F, $r_\rho = \rho_m/l\rho_f$ is the ratio of the membrane and fluid densities and s is a curvilinear coordinate along F_0. Note that the volume condition (2.8) is satisfied since the problem is solved on the fixed domain Ω_0.

In the derivation of boundary conditions (2.15) we have assumed that the tension T in the membrane is constant. However, since the membrane is not massless, the gravitational force will create a tension difference in the membrane which is of order $O(Bo)$, so that strictly speaking only small values of Bo may be considered. For convenience we will however neglect tension variations due to the mass of the membrane. A viscous flow for which the no-slip boundary condition is satisfied also creates a tension difference in the membrane due to viscous drag effects. It can, however, be shown (cf. Batchelor [14], Section 5.11) that the resulting tension difference is proportional to the square of the variation of the fluid velocity tangential to the wall. It follows that viscous drag effects are of second order and may be neglected in the linear problem.

3. Steady state deflection of F_0

In this section we shall be concerned with a $2D$ rectangular container of length L and height h, filled to the brim with a fluid. Let us, as an example of the calculation of the steady state deflection, assume that a part of the wall at $x = L$ is flexible. The treatment of other walls is analogous. Consider the case in which the wall at $x = L$ consists of a membrane, fixed at $y = 0$ and at $y = a(<h)$, and a rigid part for $a \leq y \leq h$, cf. Fig. 2. In the interval $y \in [a, h]$ the deflection is zero but on $0 \leq y \leq a$ we expect the wall to bulge under the fluid pressure exerted on the membrane. Equations (2.9)–(2.11) completely define the shape of the deflected membrane. From (2.9) and condition (2.10) we obtain

$$p_0 = \rho_f \mathbf{f}\cdot(\mathbf{x} - \mathbf{x}_{S_0}) + p_g\,,$$

which on substituting into (2.11) gives

$$\rho_f g(y - h) = \frac{T}{R_{C_0}}\,.$$

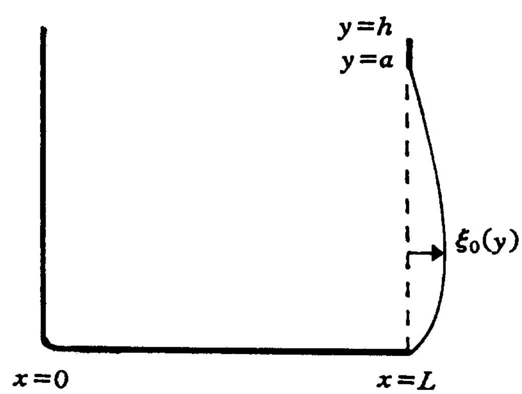

Fig. 2.

Let $\xi_0(y)$ denote the function describing the deflection of the membrane relative to the line $x = L$. Using the familiar expression for the radius of curvature we find that the above equation can be written in dimensionless form as follows

$$\frac{\xi_0''}{(1 + \xi_0'^2)^{3/2}} = \varepsilon(y - h) . \tag{3.1}$$

with boundary conditions

$$\xi_0(0) = \xi_0(a) = 0 .$$

In (3.1) the prime denotes differentiation with respect to y and we have replaced *Bo* by ε. The solution of the non-linear differential equation (3.1) can be approximated analytically when ε is small, corresponding to the case in which tension forces dominate gravitational forces. We will seek a series solution of the form

$$\xi_0(y) = \xi^{(0)}(y) + \varepsilon\xi^{(1)}(y) + \cdots ,$$

each $\xi^{(k)}(y)$ satisfying the boundary conditions at $y = 0$ and $y = a$. Substituting the above form of ξ_0 into (3.1) we find, on equating the coefficients of each power of ε to zero, that

$$\xi_0(y) = \frac{\varepsilon}{6}\,[y^3 - 3hy^2 + a(3h - a)y] + \mathrm{O}(\varepsilon^3) . \tag{3.2}$$

Note that the case $a = h$ corresponds to an entirely flexible wall.

Solving (3.1) numerically allows us to consider a wider range of values of ε. For the solution of the non-linear differential equation we use Newton's method. Let

$$F(\xi_0) = \xi_0'' - \varepsilon(y - h)(1 + \xi_0'^2)^{3/2} ,$$

so that $F(\xi_0) = 0$ is the equation to be solved subject to the two boundary conditions. Define the sequence $\xi_0^{(1)}, \xi_0^{(2)}, \ldots, \xi_0^{(n)} \to \xi_0$ by the following linear problem

$$F'(\xi_0^{(n)})[\xi_0^{(n+1)}] = F'(\xi_0^{(n)})[\xi_0^{(n)}] - F(\xi_0^{(n)}) , \tag{3.3}$$

where $F'(\phi)[\psi]$ denotes the Gateaux derivative of F defined by

$$F'(\chi)[\psi] = \lim_{\delta \to 0} \frac{1}{\delta}\,(F(\chi + \delta\psi) - F(\chi)) .$$

The linear problem defined by (3.3) can be solved using standard finite-difference techniques. Each iteration requires the solution of a system of equations, but the sequence $\xi_0^{(1)}, \xi_0^{(2)}, \ldots$ converges quadratically. In Fig. 3 we show shapes of the membrane for various values of ε, corresponding to the analytical and numerical results. Observe that the analytical and numerical results agree very well for $\varepsilon \leqslant 1$. The numerical solution procedure converges for values of ε up to $\varepsilon \approx 3.45$. Near that value of ε the derivative of ξ_0 at $y = 0$ tends to infinity. For larger values of ε a Cartesian function representation of the membrane is no

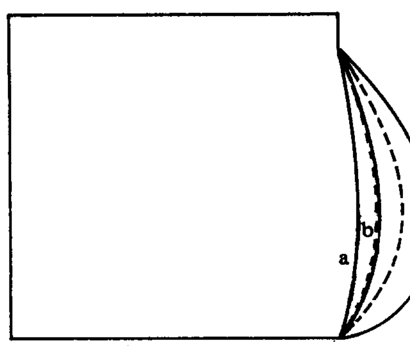

Fig. 3. Plots of the deflected membrane for $\varepsilon = 1$ (a), $\varepsilon = 2$(b) and $\varepsilon = 3.4$(c). The dotted lines are the curves given by (3.2), the solid lines correspond to numerical calculations.

longer possible since the function would be multi-valued for $y \leq 0$. A numerical technique for the calculation of the membrane deflection in that case can be found in Schulkes and Cuvelier [15].

4. Inviscid fluid oscillations

Before we proceed to solve problem (2.12)–(2.15) numerically, we investigate the related problem of an inviscid fluid oscillating in a flexible container. Under certain conditions we are able to solve this problem analytically which gives us more insight into the problem. In this section we consider the inviscid equivalent of problem (2.12)–(2.15), i.e. neglect all the terms containing the Reynolds number and replace the no-slip condition on R by the non-permeability condition. Assume that the fluid flow is initially irrotational. It is well-known that for inviscid fluids the flow remains irrotational so that we can write $\mathbf{u} = \nabla\phi$ where ϕ is a velocity potential. Assuming that p, ϕ, η and ξ exhibit a temporal behaviour of the form $e^{i\lambda t}$, it follows that equations (2.12)–(2.15) reduce to

$$\nabla^2\phi = 0 \quad \text{in } \Omega_0\,, \tag{4.1}$$

$$\frac{\partial\phi}{\partial n} = \lambda^2\phi \quad \text{on } S_0\,, \tag{4.2}$$

$$\frac{\partial\phi}{\partial n} = 0 \quad \text{on } R\,, \tag{4.3}$$

$$\frac{\partial^2}{\partial s^2}\left(\frac{\partial\phi}{\partial n}\right) + \left(\frac{1}{R_{C_0}^2} + \varepsilon(r_p\lambda^2 - \hat{\mathbf{y}}\cdot\mathbf{n}_F)\right)\frac{\partial\phi}{\partial n} = -\varepsilon\lambda^2\phi \quad \text{on } F_0\,, \tag{4.4}$$

$$\frac{\partial\phi}{\partial n} = 0 \quad \text{on } R\cap F_0\,. \tag{4.5}$$

Note that η and ξ have been eliminated using the kinematic conditions on S_0 and F_0 and that Bo has once again been replaced by ε.

We will attempt to solve (4.1)–(4.5) within the context of perturbation theory with $\varepsilon \ll 1$ as the perturbation parameter. However, the fact that the domain Ω_0 depends on ε through the shape of the wall F_0 poses a problem: functions defined on the zeroth-order domain will in general not be defined on the whole of the domain of the first order problem. This is clearly undesirable. Let us therefore assume in this section that the shape of F_0 does not depend on ε, i.e. we neglect the bending of the membrane due to the hydrostatic pressure (an *a posteriori* justification of this assumption, based on numerical results, will be given in Section 7). The boundary conditions (4.4) and (4.5) will now be written in a somewhat simplified, more convenient form. Let the membrane be fixed at the points $s = 0$ and $s = a$ (the begin and end points of F_0), conditions (4.4), (4.5) can then be shown to satisfy

$$\frac{\partial\phi}{\partial n} = \varepsilon\lambda^2\int_{F_0} K_\varepsilon(s,\theta)\phi(\theta)\,d\theta \quad \text{on } F_0\,, \tag{4.6}$$

in which

$$K_\varepsilon(s,\theta) = \begin{cases} \dfrac{1}{\kappa\tan(\kappa a)}\sin(\kappa s)[\tan(\kappa a)\cos(\kappa\theta) - \sin(\kappa\theta)]\,, & s \leq \theta \\[2ex] \dfrac{1}{\kappa\tan(\kappa a)}\sin(\kappa\theta)[\tan(\kappa a)\cos(\kappa s) - \sin(\kappa s)]\,, & s > \theta\,, \end{cases}$$

with

$$\kappa = \sqrt{\varepsilon}(r_p \lambda^2 - \hat{\mathbf{y}} \cdot \mathbf{n}_f)^{1/2} .$$

In order to solve (4.1) subject to boundary conditions (4.2), (4.3), (4.6) using perturbation techniques, we expand λ and ϕ in a power series of ε, like

$$\phi = \phi^{(0)} + \varepsilon\phi^{(1)} + \cdots , \qquad \lambda = \lambda^{(0)} + \varepsilon\lambda^{(1)} + \cdots .$$

Substituting for ϕ and λ in (4.1)–(4.3), (4.6) and equating the coefficients of like powers of ε to zero, we find the following series of problems:

$$\begin{aligned}
\varepsilon^0:\quad & \nabla^2\phi^{(0)} = 0 && \text{in } \Omega_0 , \\
& \frac{\partial\phi^{(0)}}{\partial n} = \lambda^{(0)2}\phi^{(0)} && \text{on } S_0 , \\
& \frac{\partial\phi^{(0)}}{\partial n} = 0 && \text{on } R \cup F_0 , \\
\varepsilon^1:\quad & \nabla^2\phi^{(1)} = 0 && \text{in } \Omega_0 , \\
& \frac{\partial\phi^{(1)}}{\partial n} = \lambda^{(0)2}\phi^{(1)} + 2\lambda^{(0)}\lambda^{(1)}\phi^{(0)} && \text{on } S_0 , \\
& \frac{\partial\phi^{(1)}}{\partial n} = 0 && \text{on } R , \\
& \frac{\partial\phi^{(1)}}{\partial n} = \lambda^{(0)2}\int_{F_0} K_0(s,\theta)\phi^{(0)}(\theta)\,\mathrm{d}u && \text{on } F_0 .
\end{aligned}$$

The kernel $K_0(s, \theta)$ in the last equation is given by

$$K_0(s,\theta) = \begin{cases} \dfrac{s}{a}(a-\theta) , & s \leqslant \theta \\ \dfrac{\theta}{a}(a-s) , & s > \theta . \end{cases}$$

We will now prove that the modulus of the eigenfrequencies of the fluid decreases when a part of the rigid container wall is replaced by a membrane under the aforementioned assumptions (i.e. neglecting bending due to hydrostatic pressure). We have to show that $\lambda^{(1)} < 0$ when $\lambda^{(0)} > 0$ and *vice versa*. Using Green's second identity

$$\int_{\Omega_0} (\phi^{(1)}\nabla^2\phi^{(0)} - \phi^{(0)}\nabla^2\phi^{(1)})\,\mathrm{d}x = \int_{\partial\Omega_0} \left(\phi^{(1)}\frac{\partial\phi^{(0)}}{\partial n} - \phi^{(0)}\frac{\partial\phi^{(1)}}{\partial n}\right)\mathrm{d}s ,$$

we find, on substituting the boundary conditions and employing the fact that $\phi^{(0)}$ and $\phi^{(1)}$ satisfy Laplace's equation, that $\lambda^{(1)}$ is given by

$$\lambda^{(1)} = -\tfrac{1}{2}\lambda^{(0)} \frac{\displaystyle\int_{F_0} \phi^{(0)}(s)\Big(\int_{F_0} K_0(s,\theta)\phi^{(0)}(\theta)\,\mathrm{d}\theta\Big)\,\mathrm{d}s}{\displaystyle\int_{S_0} \phi^{(0)2}\,\mathrm{d}s} . \tag{4.7}$$

The denominator in (4.7) is positive so it remains to show that the double integral in the numerator is positive. For this we have to show that the integral operator

$$K_0 f = \int_{s=0}^{s=a} K_0(s, \theta) f(\theta)\, \mathrm{d}\theta ,$$

is positive definite. The operator K_0 is self-adjoint and hence its eigenvalues are real. In appendix A it is shown that the eigenvalues of K_0 are in fact strictly positive from which it follows that the integral operator is positive definite, hence the result. Note that this result can be generalized easily to the case in which the container wall consists of a number of distinct membranes each with fixed end-points. The integral in the numerator of (4.7) is in that case replaced by the sum of the integrals over the membranes. The fact that the modulus of the eigenvalues decreases when part of the rigid container wall is replaced by a membrane, is what one might have expected on intuitive grounds since the system as a whole has more degrees of freedom and is as a result less 'rigid'.

Next we investigate the specific case of the rectangular container dealt with in the previous section, i.e. the LHS and bottom walls are rigid and the RHS wall is flexible in the interval $y \in (0, a)$ and rigid for $y \in [a, h]$. The zeroth-order problem can be solved readily, using separation of variables, to give

$$\phi_k^{(0)} = \cos \frac{k\pi x}{L} \cosh \frac{k\pi y}{L} , \qquad \lambda_k^{(0)} = \frac{k\pi}{L} \tanh \frac{k\pi h}{L} ,$$

where the subscript k refers to the kth eigenmode. Substituting $\phi_k^{(0)}$ and $\lambda_k^{(0)}$ into (4.7) yields

$$\lambda_k = \lambda_k^{(0)}\left[1 - \varepsilon \frac{a^3}{2L} \frac{1}{q^2 \cosh^2(k\pi h/L)} \left(\frac{\sinh 2q}{2q} - 1 - \frac{2}{q^2}(\cosh q - 1)^2\right)\right] , \tag{4.8}$$

in which $q = k\pi a/L$. In Figs 4a, b we show plots of $\Delta\lambda_k = (\lambda_k^{(0)} - \lambda_k)/\lambda_k^{(0)}$ versus the container length for various values of a and $k = 1, 2$. Observe first of all that the influence of

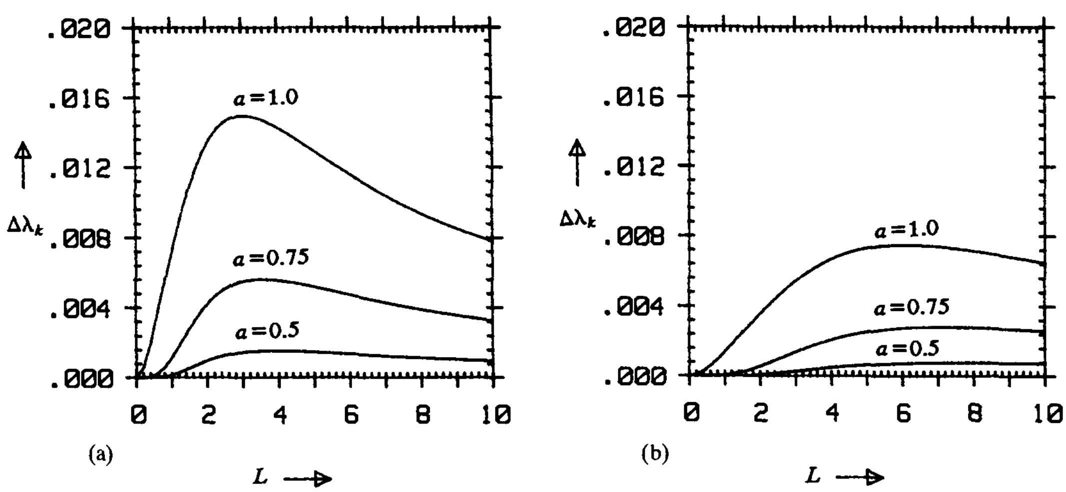

Fig. 4. Plots of $\Delta\lambda_k$ versus L for $h = 1$ and various values of a, for the first (a) and second (b) eigenmode.

the membrane on the eigenfrequencies decreases as a decreases and that the interaction of the fluid and the structure is strongest for the first eigenmode. In order to explain these observations we use the well-known fact that the fluid motion resulting from free-surface oscillations penetrates the fluid to a depth of about one wavelength (cf. Lamb [20]) and that the fluid velocity is maximal near the free surface. It is evident that as a decreases the effect of the membrane on the eigenfrequencies will decrease simply because a smaller part of the container wall is flexible. In addition however, for small values of a the membrane will be in contact with a region of fluid in which relatively low pressure gradients occur (as a result of the low fluid velocity) so that the fluid is capable of deflecting the membrane only slightly. The change of the eigenfrequencies of the fluid will be correspondingly small. A similar argument holds in order to explain the observation that the interaction is strongest for the first mode and weaker for subsequent modes.

Interesting is the observation that $\Delta\lambda$ has a well-defined maximum for some value of L. The position of the maximum and its height are dependent on both a and k: for $a = 1.0$, $k = 1$ the maximum occurs at $L = 3.02$ while for $a = 1.0$, $k = 2$ the maximum is situated at $L = 6.03$. It is easy to show that $\Delta\lambda_k$ tends to zero in the limits $q \to 0$ and $q \to \infty$. A remark is in place regarding the results presented in Figs 4a, b. We have taken $h = 1$ and calculated $\Delta\lambda_k$ for container lengths up to $L = 9$. The linear theory of small free surface deflections is, however, valid only when the elevation of the free surface is small compared with the depth of the fluid. Hence, the results in Figs 4a, b for L greater than approximately 3 should be treated with caution. However, we do expect $\Delta\lambda_k$ to decrease as L increases because the potential energy of the fluid increases (proportional to L) while the potential energy of the membrane remains constant if we increase L. Thus the effect of the membrane on the fluid motion will decrease.

We conclude this section by considering the case in which the side walls of the rectangular container are rigid but the entire bottom wall consists of a membrane. The solution of the zeroth-order problem is as in the previous case. Substituting for $\phi_k^{(0)}$ and $\lambda_k^{(0)}$ in (4.7), where the integrals in the numerator are now over the bottom wall, we obtain

$$\lambda_k = \lambda_k^{(0)}\left[1 - \frac{\varepsilon}{2}\left(\frac{L}{n\pi}\right)^2 \frac{1}{\cosh^2(n\pi h/L)}\left(1 - \frac{2}{(n\pi)^2}(1-(-1)^n)^2\right)\right].$$

Note that $\Delta\lambda_k \to 0$ as $L \to 0$, but unlike the previous case $\Delta\lambda_k \propto L^2$ for large L, i.e. the interaction between fluid and membrane motion increases as L increases. This may be explained by the fact that the total energy of both the fluid and the membrane increases as L increases.

5. Viscous oscillations: properties of the spectrum

Equations (2.12)–(2.15) will be solved numerically using a finite element approach. To that end we write (2.12)–(2.15) in a variational formulation for which we use the identity (cf. Cuvelier et al. [21]),

$$\int_{\Omega_0}\left(\frac{\partial \mathbf{u}}{\partial t} + \nabla p - \frac{1}{Re}\nabla^2\mathbf{u}\right)\cdot \mathbf{v}\,\mathrm{d}x = \int_{\Omega_0}\left(\frac{\partial \mathbf{u}}{\partial t}\cdot \mathbf{v} - p\nabla\cdot\mathbf{v}\right)\mathrm{d}x$$

$$+ \frac{1}{Re}a(\mathbf{u}, \mathbf{v}) - \int_{\partial\Omega_0}(\sigma_n v_n + \sigma_\tau v_\tau)\,\mathrm{d}s\ ,$$

in which

$$a(\mathbf{u},\mathbf{v})=\tfrac{1}{2}\sum_{i,j=1}^{2}\int_{\Omega_0}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right)\mathrm{d}x\,,\quad x_1=x,\quad x_2=y\,.$$

Let us assume that $\mathbf{u}$, p, η and ξ exhibit a temporal behaviour of the form $\exp(\mu t)$ in which μ is in general a complex quantity. The variational formulation of equations (2.12) becomes on employing the above identity and the boundary conditions (2.13)–(2.15):

find $\mathbf{u}$ and p such that for all suitably smooth functions $\mathbf{v}$ and q

$$\int_{\Omega_0}(\mu\mathbf{u}\cdot\mathbf{v}-p\nabla\cdot\mathbf{v})\,\mathrm{d}x+\frac{1}{Re}\,a(\mathbf{u},\mathbf{v})+\int_{S_0}\frac{1}{\mu}\,u_nv_n\,\mathrm{d}s \tag{5.1}$$

$$+\int_{F_0}\left[\mu r_\rho u_nv_n+\frac{1}{\mu}\,u_nv_n\hat{\mathbf{y}}\cdot\mathbf{n}_F+\frac{1}{\mu\,Bo}\left(\frac{\partial u_n}{\partial s}\,\frac{\partial v_n}{\partial s}-\frac{u_nv_n}{R^2_{C_0}}\right)\right]\mathrm{d}s=0\,,$$

$$\int_{\Omega_0}q\nabla\cdot\mathbf{u}\,\mathrm{d}x=0\,, \tag{5.2}$$

where $\mathbf{u}$ and p are in general complex functions. It is important to note that all degrees of freedom related to the membrane have been eliminated from the problem by virtue of the kinematic condition. We are left with a problem in which only quantities related to the fluid (fluid velocity and pressure) are unknowns. This type of problem is considerably easier to deal with than the one in which both fluid and structure unknowns are present.

Let us now investigate the spectrum of the problem defined by equations (5.1) and (5.2). To that end we consider the quadratic functional defined by

$$\Phi=\mu^2((\mathbf{u},\mathbf{u})_{\Omega_0}+r_\rho(u_n,u_n)_{F_0})+\mu\,\frac{1}{Re}\,a(\mathbf{u},\mathbf{u})+(u_n,u_n)_{S_0}+\frac{1}{Bo}\,b(u_n,u_n)\,.$$

The functional Φ takes on a stationary value (equal to zero) when $\mathbf{u}$ is a solution of equations (5.1), (5.2). We have introduced the notation

$$(\mathbf{u},\mathbf{u})_{\Omega_0}=\int_{\Omega_0}|\mathbf{u}|^2\,\mathrm{d}x,\ \ldots\ \text{ etc.}\,,$$

$$b(u_n,u_n)=\int_{F_0}\left(u_nu_n\left(Bo\,\hat{\mathbf{y}}\cdot\mathbf{n}_F-\frac{1}{R^2_{C_0}}\right)+\frac{\partial u_n}{\partial s}\,\frac{\partial u_n}{\partial s}\right)\mathrm{d}s\,.$$

Set Φ equal to zero and treat the resulting equation as a quadratic in μ. Solving for μ we arrive at the following qualitative properties concerning vibration modes of a viscous fluid in an elastic container:

(i) The problem as defined by (5.1), (5.2) is stable with respect to infinitesimal perturbations when

$$(u_n,u_n)_{S_0}+(1/Bo)b(u_n,u_n)>0\,;$$

(ii) If

$$\frac{1}{Re^2}\,(a(\mathbf{u},\mathbf{u}))^2\geqslant 4((\mathbf{u},\mathbf{u})_{\Omega_0}+r_\rho(u_n,u_n)_{F_0})\Big((u_n,u_n)_{S_0}+\frac{1}{Bo}\,b(u_n,u_n)\Big)\,,$$

then the eigenvalues corresponding to the eigenfunctions $\mathbf{u}$ are real and negative. This corresponds to an aperiodic damping process;

(iii) If

$$\frac{1}{Re^2}(a(\mathbf{u},\mathbf{u}))^2 < 4((\mathbf{u},\mathbf{u})_{\Omega_0} + r_\rho(u_n, u_n)_{F_0})\Big((u_n, u_n)_{S_0} + \frac{1}{Bo}\, b(u_n, u_n)\Big),$$

then the eigenvalues are complex occurring in complex conjugate pairs. The real part of the eignvalues is negative so that oscillations are damped.

It is easy to show that the stability condition in (i) is satisfied for sufficiently small values of Bo (by virtue of the fact that $1/R_{C_0}$ and $\hat{\mathbf{y}} \cdot \mathbf{n}_F \sim Bo$, cf. Section 3). The stability requirement stems from the fact that the membrane is taken to be perfectly elastic and is physically analogous to the stability of a pendant liquid drop: for a certain value of the surface tension and strength of the gravitational field (essentially the Bond number) there is a maximum drop size – larger drops are not stable.

Let us consider the expression for μ in some more detail when Re is large. Retaining only terms to first order in $1/Re$ we obtain

$$\mu \approx \frac{-a(\mathbf{u},\mathbf{u})}{2Re[(\mathbf{u},\mathbf{u})_{\Omega_0} + r_\rho(u_n, u_n)_{F_0}]} \pm \mathrm{i}\left(\frac{(u_n, u_n)_{S_0} + \frac{1}{Bo}\, b(u_n, u_n)}{(\mathbf{u},\mathbf{u})_{\Omega_0} + r_\rho(u_n, u_n)_{F_0}}\right)^{1/2}. \tag{5.3}$$

Now assume that for a certain set of parameter values eigenfunctions $\mathbf{u}$ exist such that $\mathbf{u}$ on F_0 is small compared with $\mathbf{u}$ on S_0. Since the terms containing Bo and r_ρ in (5.3) are quadratic in $\mathbf{u}$ on F_0 they may be neglected in this case. Hence equation (5.3) reduces to

$$\mu \approx \frac{-a(\mathbf{u},\mathbf{u})}{2Re(\mathbf{u},\mathbf{u})_{\Omega_0}} \pm \mathrm{i}\left(\frac{(u_n, u_n)_{S_0}}{(\mathbf{u},\mathbf{u})_{\Omega_0}}\right)^{1/2}. \tag{5.4}$$

Equation (5.4) is the familiar expression for the eigenfrequencies of weakly damped free-surface oscillations of a viscous fluid in a rigid container expressed in terms of the corresponding eigenmodes, see for example Kopachevskii and Myshkis [16]. In sequel these modes will be termed free-surface oscillations mode.

Assume next that eigenfunctions exist such that $\mathbf{u}$ on S_0 is much smaller than on F_0. The term related to free surface oscillations in the denominator of $\mathrm{Im}(\mu)$ in (5.3) may then be neglected giving

$$\mu \approx \frac{-a(\mathbf{u},\mathbf{u})}{2Re[(\mathbf{u},\mathbf{u})_{\Omega_0} + r_\rho(u_n, u_n)_{F_0}]} \pm \mathrm{i}\left(\frac{\frac{1}{Bo}\, b(u_n, u_n)}{(\mathbf{u},\mathbf{u})_{\Omega_0} + r_\rho(u_n, u_n)_{F_0}}\right)^{1/2}. \tag{5.5}$$

Observe that $\mathrm{Im}(\mu)$ in (5.5) is proportional to $1/\sqrt{Bo}$. The functional dependence on Bo is in fact the same for a freely vibrating membrane (recall that eigenfrequencies of a membrane are proportional to the square root of the tension). We conclude, therefore, that eigenmodes for which $\mathbf{u}$ is small on S_0 are associated with normal mode oscillations of the membrane. These modes will be called structural-vibration modes. Observe that $\mathrm{Re}(\mu)$ in (5.5) contains an extra term in the denominator as compared with $\mathrm{Re}(\mu)$ in (5.4). This indicates that the structural-vibration modes may be damped less than the free-surface oscillation modes.

Of interest is the term representing the inertia of the fluid, $(\mathbf{u},\mathbf{u})_{\Omega_0}$, in the denominator of $\mathrm{Im}(\mu)$ in (5.5). It is not hard to show that the eigenfrequencies of a freely vibrating membrane in terms of the eigenfunctions are given by the expression $\mathrm{Im}(\mu)$ in (5.5), however, without the fluid inertia term ($\mathrm{Im}(\mu)$ is in that case just the Rayleigh quotient where u_n corresponds to the deflection of the membrane). It follows that the eigenfrequencies of a membrane in contact with a fluid will be less than those of a freely vibrating membrane. It appears, therefore, that the effect of the fluid is essentially to increase the mass of the membrane. This observation is related to the concept of added-mass. Namely, when solving fluid-structure interaction problems numerically one can, in certain cases, eliminate the unknowns related to the fluid motion by adding an "added-mass" matrix to the mass matrix of the dry structure, refer to for example Deruntz and Geers [17] and Muller [18].

6. The discrete eigenvalue problem

Although the finite-element discretization technique can be applied directly to equations (5.1) and (5.2), the presence of the incompressibility constraint leads to a system of equations which has unfavourable properties from a numerical point of view. Namely, either the bandwidth of one of the matrices is large or a partial pivoting procedure has to be applied, both leading to a substantial increase in computing time. These problems can be eliminated when the continuity equation is perturbed by a small (penalty) parameter times the pressure, viz.

$$\varepsilon_p p + \nabla \cdot \mathbf{u} = 0 \,,$$

or in variational formulation

$$\varepsilon_p \int_{\Omega_0} pq \, \mathrm{d}x + \int_{\Omega_0} q \nabla \cdot \mathbf{u} \, \mathrm{d}x = 0 \,, \tag{6.1}$$

where $\varepsilon_p \ll 1$ (usually $\varepsilon_p \sim 10^{-6}$). Dividing Ω_0 up into triangles and writing the velocity $\mathbf{u}$ as a linear combination of extended quadratic basis functions and the pressure p as a linear combination of linear basis functions (cf. Cuvelier et al. [21]), we obtain the discrete equivalents of (5.1) and (6.1), namely

$$\mu \mathbf{M}\hat{\mathbf{u}} - \mathbf{L}^T\hat{\mathbf{p}} + \frac{1}{Re}\mathbf{A}\hat{\mathbf{u}} + \frac{1}{\mu}\mathbf{B}_{S_0}\hat{\mathbf{u}} + \frac{1}{\mu}\mathbf{B}_{F_0}\hat{\mathbf{u}} + \mu \mathbf{M}_{F_0}\hat{\mathbf{u}} = 0 \,,$$

$$\varepsilon_p \mathbf{D}\hat{\mathbf{p}} = -\mathbf{L}\hat{\mathbf{u}} \,.$$

The complex vectors $\hat{\mathbf{u}}$ and $\hat{\mathbf{p}}$ contain the velocity and pressure unknowns respectively in the nodal points. Eliminating the pressure from the above equations we obtain an eigenvalue problem of the form

$$(\mu^2 \mathbf{M}_T + \mu \mathbf{S} + \mathbf{B}_T)\hat{\mathbf{u}} = 0 \,, \tag{6.2}$$

in which

$$\mathbf{M}_T = \mathbf{M} + \mathbf{M}_{F_0}\,, \qquad \mathbf{S} = \frac{1}{\varepsilon_p}\,\mathbf{L}^T\mathbf{D}^{-1}\mathbf{L} + \frac{1}{Re}\,\mathbf{A}\,, \qquad \mathbf{B}_T = \mathbf{B}_{S_0} + \mathbf{B}_{F_0}\,.$$

All matrices are real and symmetric, and $\mathbf{M}_T$, $\mathbf{S}$ are positive definite. $\mathbf{M}$ and $\mathbf{M}_{F_0}$ are mass matrices corresponding to the fluid and membrane motions respectively; $\mathbf{M}_{F_0}$ is singular. $\mathbf{B}_{S_0}$ and $\mathbf{B}_{F_0}$ are matrices representing the potential energies of the perturbed free surface and membrane respectively and are both singular since only the degrees of freedom corresponding to the free surface and the membrane give non-zero entries.

In order to investigate the properties of the discrete eigenvalue problem we write (6.2) in the form of standard eigenvalue problem, viz.

$$\begin{pmatrix} 0 & \mathbf{I} \\ -\mathbf{M}_T^{-1}\mathbf{B}_T & -\mathbf{M}_T^{-1}\mathbf{S} \end{pmatrix}\begin{pmatrix} \hat{\mathbf{u}} \\ \hat{\mathbf{v}} \end{pmatrix} = \mu\begin{pmatrix} \hat{\mathbf{u}} \\ \hat{\mathbf{v}} \end{pmatrix} \tag{6.3}$$

where the substitution $\mu\hat{\mathbf{u}} = \hat{\mathbf{v}}$ has been made. Consider the case in which the region Ω_0 has been discretized in such a way that on the free surface we have n_S degrees of freedom, on the membrane we have n_F degrees of freedom and the total number of degrees of freedom is equal to $n > n_S + n_F$. It follows that $\operatorname{rank}(\mathbf{M}_T) = \operatorname{rank}(\mathbf{S}) = n$ and $\operatorname{rank}(\mathbf{B}_T) = \operatorname{rank}(\mathbf{B}_{S_0}) + \operatorname{rank}(\mathbf{B}_{F_0}) = n_S + n_F$. Hence, the large matrix in (6.3) is of order $2n$ with rank equal to $\operatorname{rank}(\mathbf{B}_T) + \operatorname{rank}(\mathbf{S}) = n + n_S + n_F$. It follows immediately that $2n - (n + n_S + n_F) = n - n_S - n_F$ of the eigenvalues of (6.3) are zero. The corresponding eigenvectors are of the form $(\hat{\mathbf{u}}, 0)^T$ with $\hat{\mathbf{u}} \in N(\mathbf{B}_T)$ where $N(\mathbf{B}_T)$ denotes the null-space of $\mathbf{B}_T$. Next consider the case in which $\hat{\mathbf{u}} \in N(\mathbf{B}_T)$ and $\mu \neq 0$. From (6.3) we obtain

$$\hat{\mathbf{v}} = \mu\hat{\mathbf{u}}\,, \tag{6.4}$$

$$(\mu\mathbf{M}_T + \mathbf{S})\hat{\mathbf{u}} = 0\,. \tag{6.5}$$

It follows that eigenvectors of (6.3) exist with $\hat{\mathbf{u}} \in N(\mathbf{B}_T)$ and $\mu \neq 0$ if $\hat{\mathbf{v}}$ is an eigenvector of (6.5) with corresponding eigenvector μ and, moreover, $\hat{\mathbf{v}} \in N(\mathbf{B}_T)$ as follows from (6.4). Since the matrices $\mathbf{M}_T$ and $\mathbf{S}$ are positive definite it follows that (6.5) has a complete set of eigenvectors and the corresponding eigenvalues are real and negative. Since $N(\mathbf{B}_T)$ has order $n - n_S - n_F$ it follows that $n - n_S - n_F$ linearly independent eigenvectors of (6.5) are in $N(\mathbf{B}_T)$. Thus (6.3) has $n - n_S - n_F$ real and negative eigenvalues with corresponding eigenvectors of the form $(\hat{\mathbf{u}}, \hat{\mathbf{v}})$ with $\hat{\mathbf{u}}, \hat{\mathbf{v}} \in N(\mathbf{B}_T)$. We conclude that the eigenvalue problem (6.2) has $2n$ eigenvalues of which $n - n_S - n_F$ are zero and $n - n_S - n_F$ are real and negative. The remaining $2n_S + 2n_F$ eigenvalues correspond to the free-surface modes ($2n_S$ eigenvalues) and structural-vibration modes ($2n_F$ eigenvalues) occurring in complex conjugate pairs. The properties of the eigenmodes formulated in the previous section are related to these eigenvalues.

The quadratic eigenvalue problem (6.2) is of the type often encountered when dealing with structural vibrations. In general one is interested only in the eigenmodes corresponding to the complex eigenvalues. Of those eigenmodes, the ones corresponding to eigenfrequencies with the smallest imaginary parts are generally of principal engineering interest. Calculating the whole spectrum would then be a wasteful computational exercise. Schulkes and Cuvelier [13] found that if only a few eigenmodes are of interest, then these can be calculated efficiently using an inverse iteration procedure. This procedure will also be used in this paper, for details the reader is referred to the aforementioned paper.

7. Numerical results

We start our numerical experiment with a comparison of numerical results and the analytical results of Section 4. To that end we consider the container dealt with in Section 3 with $h = 1$, $a = 0.9$, $Bo = 0.1$ and assume that the container is entirely filled with a viscous fluid with $Re = 5 \times 10^2$. The region Ω_0 is discretized by 504 triangular elements, refined at the boundaries in order to capture the boundary layer. In the numerical calculation the bending of the membrane due to the hydrostatic pressure is not neglected so that the shape of F_0 is given by the solution of (3.1). Figure 5 shows a plot of $\Delta\,\mathrm{Im}(\mu_1)$ versus the container length L, where $\Delta\,\mathrm{Im}(\mu_1)$ is given by

$$\Delta\,\mathrm{Im}(\mu_1) = \frac{\mathrm{Im}(\mu_r) - \mathrm{Im}(\mu_1)}{\mathrm{Im}(\mu_r)} ,$$

in which μ_1 is the first eigenmode of the fluid and μ_r the corresponding mode in a similarly shaped rigid container. We have also plotted the curve $\Delta\lambda_1$ versus L where $\Delta\lambda_1$ is obtained from (4.8). Observe that, even though viscous effects and bending of the membrane are neglected in the analytical calculation, there is a close agreement between the numerical and analytical results. It follows that for small Bond numbers the bending of the membrane due to the hydrostatic pressure may be neglected (cf. the assumptions made in the analysis of Section 4).

As was pointed out in Section 5, we expect to observe at least two different types of normal oscillation modes namely, free-surface modes and the structural-vibration modes. We now investigate some of the characteristics of these modes. We consider the same container as in the previous experiment with $L = 1$ and choose the parameter values $Re = 5 \times 10^2$, $Bo = 0.2$ and $r_p = 1$. In Fig. 6 vector plots of the fluid velocity are shown of the first two free-surface modes (6a, b) and the first and second structural-vibration modes (6c, d). Observe that, for the present parameter values, the membrane is hardly deflected in the case

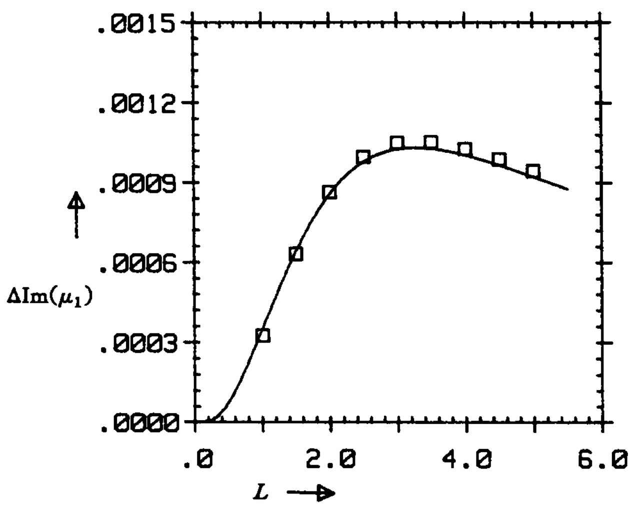

Fig. 5. Plots of $\Delta\,\mathrm{Im}(\mu_1)$ and $\Delta\lambda_1$ versus L. The solid line is the analytical curve for $\Delta\lambda_1$ obtained from (4.8), the values of $\Delta\,\mathrm{Im}(\mu_1)$ are denoted by a □.

of the free-surface modes. In the case of the principal structural-vibration mode the membrane is deflected significantly causing the free surface to be displaced as well. In the case of the second structural-vibration mode the membrane is deflected significantly while the free surface is hardly effected. The calculated eigenvalues corresponding to both types of oscillation modes are shown in Table 1. Note that the damping coefficients of the structural-vibration modes are significantly less than those of the free-surface modes corresponding to the same modal number. The number in brackets behind the eigenfrequencies of the first and second structural vibration modes in Table 1, are the first and second eigenfrequencies respectively of a freely vibrating membrane. We observe that the inertia of the fluid lowers the imaginary part of the eigenfrequencies. These results are in accordance with the qualitative remarks made in Section 5.

We next investigate how eigenfrequencies corresponding to the free-surface and structural-vibration modes vary as a function of the Bond number. Since the functional dependence is, as we shall find, critically dependent on the Reynolds number, we perform the calculations for two representative values of the Reynolds number, namely $Re = 5 \times 10^2$ and $Re = 5 \times 10^3$. The container to be considered is as in the previous calculations with $L = 1$, $a = 0.9$ and $r_p = 1$. We restrict our investigation to the dependence on Bo of the second free-surface mode (denoted by μ_f) and the first structural-vibration mode (denoted by μ_s). The Bond number is taken in the range $0.9 \leqslant Bo \leqslant 1.6$. In Fig. 7a we have plotted $\mathrm{Im}(\mu_f)$ (the dotted line) and $\mathrm{Im}(\mu_s)$ (the drawn line) versus Bo for the case $Re = 5 \times 10^2$. Observe that $\mathrm{Im}(\mu_f)$ is virtually independent of Bo. In fact $\mathrm{Im}(\mu_f) \approx 2.5$ which is equal to the imaginary part of the second free-surface mode in a rigid rectangular container with the same dimensions. The drawn line in Fig. 7a is a curve proportional to $1/\sqrt{Bo}$ and intersects the curve of $\mathrm{Im}(\mu_f)$ near $Bo \approx 1.2$. The dependence of the real parts of μ_s and μ_f on Bo is shown in Fig. 7b. Note that $\mathrm{Re}(\mu_f)$ and $\mathrm{Re}(\mu_s)$ do not change significantly with Bo and that $\mathrm{Re}(\mu_f) \gg \mathrm{Re}(\mu_s)$.

Let us next consider the case in which $Re = 5 \times 10^3$. In Fig. 8a we have plotted $\mathrm{Im}(\mu)$ corresponding to the free-surface and structural-vibration modes as a function of Bo. Figure 8b shows the dependence on Bo of the corresponding values of $\mathrm{Re}(\mu)$. We observe that for Bo up to $Bo \approx 1.2$ the dotted curve behaves quite similar to the dotted curve in Fig. 7a, i.e. it represents the free-surface mode which is independent of Bo. However, near $Bo \approx 1.2$ (the point at which the curves in Fig. 7a intersect) the dotted curve starts to decrease and its shape starts to resemble that of the drawn curve in Fig. 7a. This suggests that the free-surface mode has changed over into a structural-vibration mode. Inspection of the corresponding eigenfunctions reveals that this is indeed the case. Regarding the drawn line in Fig. 8a we observe the opposite behaviour: it first represents structural-vibration modes but changes over into a branch representing free-surface modes after $Bo \approx 1.2$. When we consider $\mathrm{Re}(\mu)$ of the corresponding modes (Fig. 8b), we observe a rapid decrease in $\mathrm{Re}(\mu)$ as the free-surface mode changes over into a structural vibration mode (dotted line) and the opposite behaviour when the structural mode changes over into a free-surface mode (drawn

Table 1.

μ	Type of mode
$-0.097 + 1.76i$	First free-surface mode
$-0.21 + 2.49i$	Second free-surface mode
$-0.038 + 6.20i$ (7.81)	First structural-vibration mode
$-0.052 + 14.1i$ (15.61)	Second structural-vibration mode

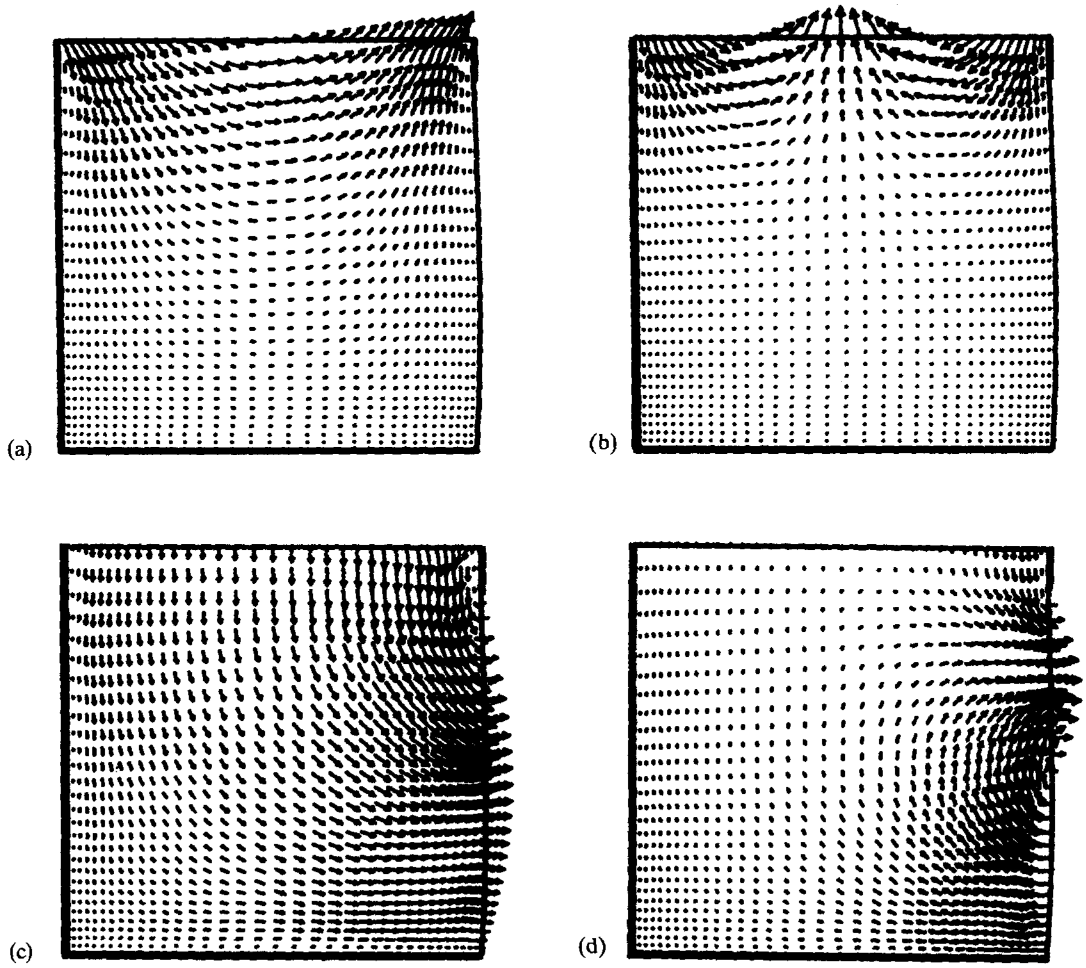

Fig. 6. Velocity vector plots of the first (a) and second (b) free-surface modes and the first (c) and second (d) structural-vibration modes.

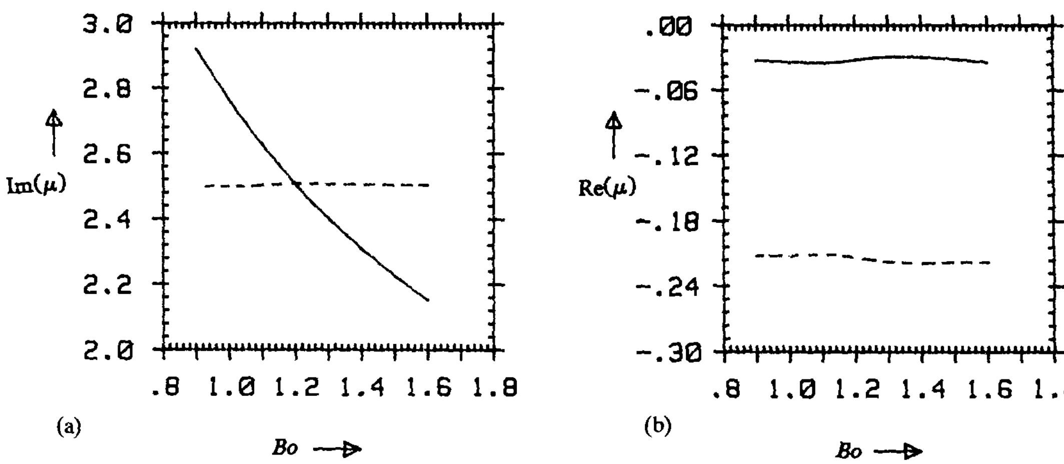

Fig. 7. Figure 7a shows the dependence on *Bo* of $\mathrm{Im}(\mu_f)$ (indicated by the dotted line) and $\mathrm{Im}(\mu_s)$ (indicated by the drawn line). The curves $\mathrm{Re}(\mu_f)$ (dotted line) and $\mathrm{Re}(\mu_s)$ (drawn line) versus *Bo* are shown in Fig. 7b. Calculations are for the case $Re = 5 \times 10^2$.

line). The qualitative statements of Section 5 regarding free-surface and structural-vibration modes are, apart from the behaviour in the transition region near $Bo = 1.2$, still valid. Namely, free-surface modes are independent of Bo and damped more than the structural-vibration modes, the structural-vibration modes display a $1/\sqrt{Bo}$ behaviour.

The bifurcation behaviour of the curves in Fig. 8a is typical for the case when branches, representing two types of eigenmodes of a system without damping, meet for a certain parameter value. See for example Santini and Barboni [19] who consider the normal modes of a system consisting of an inviscid fluid and a translationally or rotationally moveable wall. We find that the bifurcation behaviour is not removed when damping is included (Fig. 8a, high Reynolds number) but only when damping is sufficiently high (Fig. 7a, low Reynolds number). Note that in the complex plane the branches corresponding to free-surface and structural-vibration modes always remain separated since for no value of Bo are the real and imaginary parts of the two modes identical.

Streamlines of the eigenfunctions corresponding to eigenmodes in the transition region near $Bo = 1.2$ are displayed in Fig. 9. The streamlines in Figs 9a, b, c correspond to the free-surface modes for $Bo = 1.1$, 1.2 and 1.3 respectively and $Re = 5 \times 10^2$, i.e. the imaginary parts of the corresponding eigenfrequencies lie on the dotted line in Fig. 7a. The streamlines in Figs 9a and b indicate a significant displacement of the free-surface and the membrane. In Fig. 9c we observe a pattern characteristic of the second free-surface mode with no significant displacement of the membrane (cf. Fig. 6b). It follows that the term 'free-surface mode' is not quite justified when the eigenfrequencies of the fluid and structure are close. Namely, both the free surface and the membrane experience modal displacements. It is in this case more appropriate to speak of 'coupled modes'. Next consider the streamlines in Figs 9d, e, f. They correspond to parameter values $Bo = 1.1$, 1.2 and 1.3 and $Re = 2 \times 10^3$. The streamlines are of eigenfunctions of which the imaginary parts of the corresponding eigenfrequencies lie on the dotted line in Fig. 8a. The streamline pattern in Fig. 9d is characteristic of the second free-surface mode, however, displaying some interaction with the membrane. The pattern in Fig. 9e is, as Figs 9a and b, a typical coupled-mode pattern: the

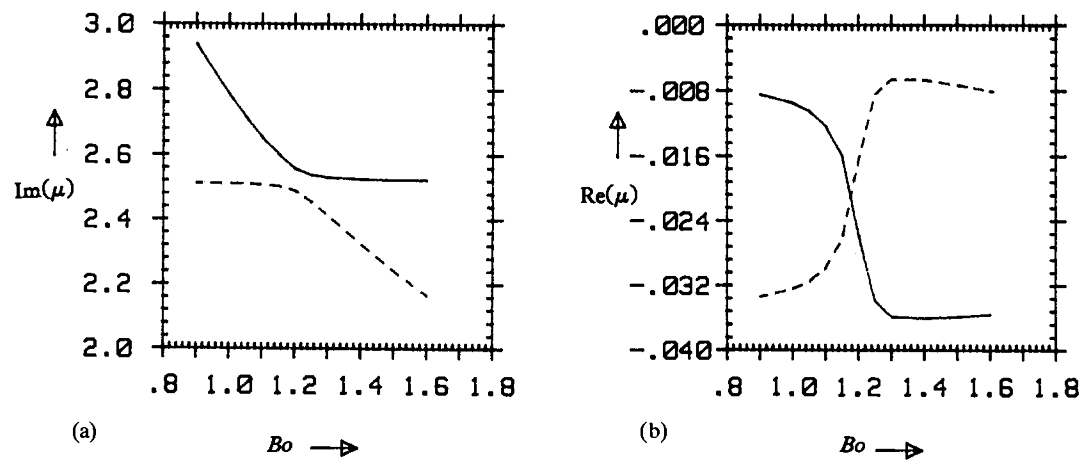

Fig. 8. Figures 8a and b show the dependence on Bo of the imaginary and real parts respectively of the eigenfrequencies corresponding to the free-surface and first structural-vibration mode for $Re = 5 \times 10^3$. The imaginary part indicated by the dotted line in 8a corresponds to the real part indicated by the dotted line in 8b and likewise for the drawn lines.

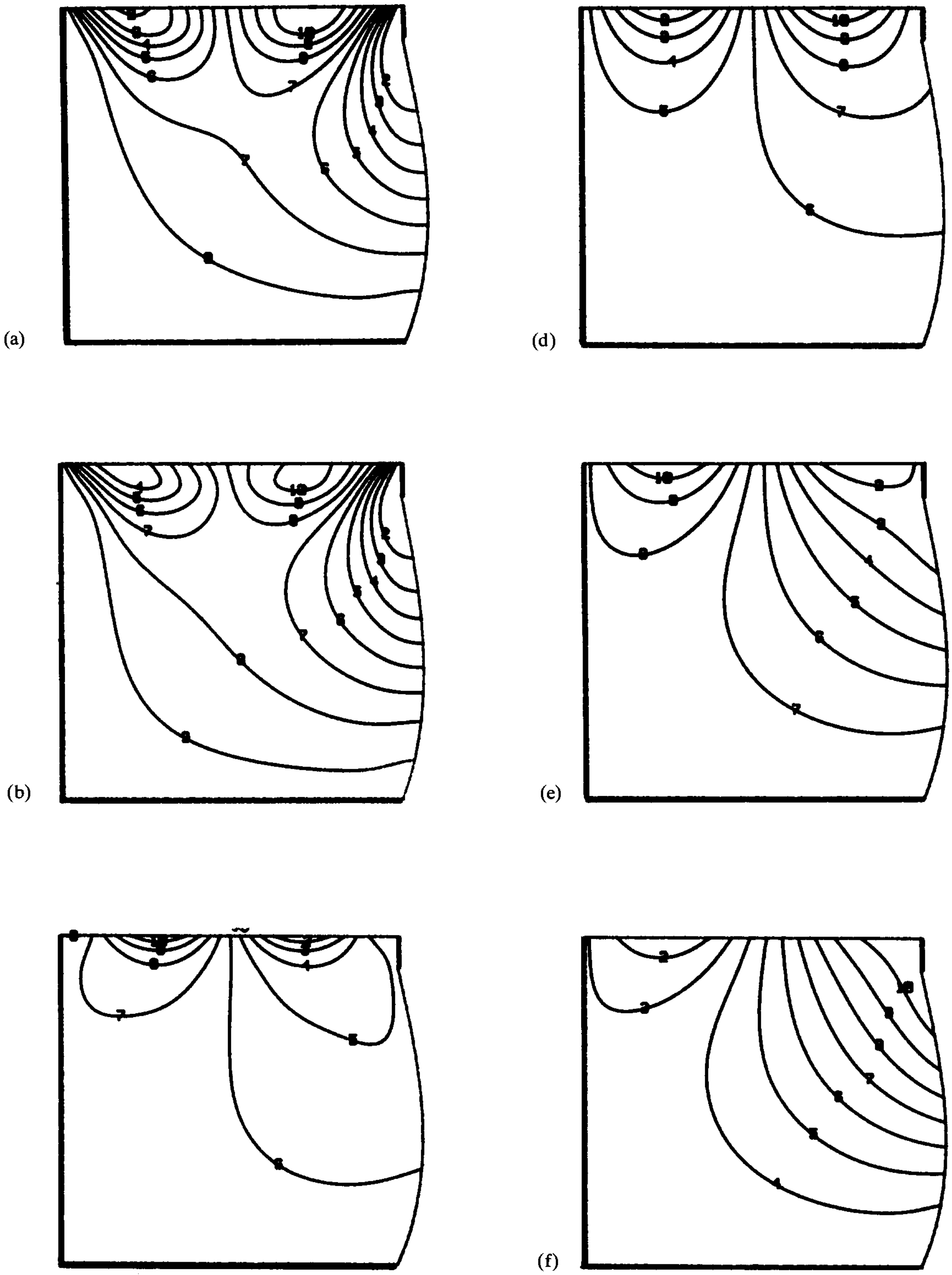

Fig. 9. Streamlines of the fluid flow. Figures 9b, c correspond to the parameter values $Bo = 1.1$, $Bo = 1.2$ and $Bo = 1.3$ respectively with $Re = 5 \times 10^2$; the imaginary parts of the eigenmodes lie on the dotted line in Fig. 7a. Figures 9d, e, f correspond to the same Bond numbers as 9a, b, c respectively but with $Re = 5 \times 10^3$; the imaginary parts of the eigenmodes lie on the dotted line in Fig. 8a.

Table 2.

	First mode		Second mode	
Re	$Bo = 0.1$	$Bo = 1.0$	$Bo = 0.1$	$Bo = 1.0$
1000	$-0.029 + 8.78i$	$-0.025 + 2.77i$	$-0.037 + 20.0i$	$-0.026 + 6.25i$
100	$-0.11 + 8.71i$	$-0.070 + 2.73i$	$-0.17 + 19.9i$	$-0.10 + 6.20i$
10	$-0.41 + 8.52i$	$-0.30 + 2.64i$	$-0.76 + 19.7i$	$-0.57 + 6.08i$
1	$-2.8 + 7.92i$	$-2.1 + 0i$	$-5.6 + 18.6i$	$-5.3 + 3.09i$

free surface and membrane are displaced significantly. In Fig. 9f we see a pattern which resembles that of the first structural-vibration mode (cf. Fig. 6c). Investigating the streamline patterns of the normal modes corresponding to values of *Bo* sufficiently far removed from the value $Bo = 1.2$, we find that they reduce to the characteristic patterns of the free-surface and structural vibration modes.

The final aspect of the fluid-structure interaction problem we will investigate, is the dependence of the structural-vibration mode on the viscosity of the fluid. In Table 2 we have listed the eigenfrequencies of the first and second structural-vibration modes for various values of the Reynolds number and two different Bond numbers. Observe that for $Bo = 0.1$ the damping coefficients increase by two orders of magnitude as *Re* goes from 1000 to 1 while the imaginary parts decrease by only 10%. For $Bo = 1.0$ we see the same increase in the damping coefficients but the decrease in the imaginary parts is considerably more than for $Bo = 0.1$. For $Bo = 1.0$ and $Re = 1.0$ the imaginary part of the first mode, in fact, completely vanishes. In order to explain this observation we go back to property (ii) of the spectrum as formulated in Section 5. Note that, in relation to the results presented in Table 2, we expect the inequality in (ii) to be satisfied for any sufficiently small value of *Re*. However, small values of *Bo* require accordingly smaller values of *Re* to satisfy the inequality which is in agreement with the calculations.

8. Conclusions

In this paper we have studied the normal modes of a system consisting of an open vessel with a flexible wall containing a viscous, incompressible fluid. We have shown that the eigenfrequencies related to free-surface oscillation modes of an inviscid fluid in a container decrease when part of the rigid container wall is replaced by a membrane. From the expression for the eigenvalues of a weakly damped viscous fluid in a flexible container we are able to deduce a number of qualitative results. Namely, if normal modes exist such that the membrane is not displaced significantly, then the eigenfrequencies of this mode are approximately the same as the eigenfrequencies of a viscous fluid in a similarly shaped rigid container. These modes are termed free-surface oscillation modes and are independent of the Bond number. On the other hand, if modes exist such that the free surface is displaced only slightly then the imaginary parts of the corresponding eigenfrequencies are proportional to $1/\sqrt{Bo}$. These modes are termed structural vibration modes, their eigenfrequencies are less than those of a freely vibrating membrane. Structural vibrations may experience less damping than the free surface oscillations.

Numerically we find that the results obtained for an inviscid fluid are also true when viscous effects are included. We find, furthermore, that free-surface and structural-vibration modes indeed exist. The properties of these modes agree well with the qualitative analytical results mentioned above. Of interest is the case in which an eigenfrequency of the fluid is close to an eigenfrequency of the membrane, and in particular its dependence on the Bond number. We find that the Reynolds number is an important parameter in this case. For small values of *Re* the eigenfrequencies of the free-surface modes are found to be independent of the Bond number, while the imaginary parts of the eigenfrequencies of the structural-vibration modes are proportional to the reciprocal of the square root of *Bo*. For a certain 'critical' value of *Bo* the imaginary parts of both modes coincide. For large values of *Re* the graphs of the imaginary parts of the two different modes are globally the same as in the case of small *Re*. However, near the aforementioned critical value of *Bo* to two curves display a bifurcation-type of behaviour. Namely, for a particular value of *Bo* the two different branches correspond to either the free-surface of structural-vibration modes, increasing *Bo* will cause the branch representing free-surface modes to change over into a branch representing structural-vibration modes and *vice versa*. The changeover occurs in the neighbourhood of the critical value of *Bo*. In the vicinity of the critical value of *Bo* the normal modes of the fluid and the membrane are coupled, i.e. simultaneous free-surface and membrane deflections are observed, both closely resembling the corresponding modal deflections in the uncoupled problem.

Appendix A

Consider the operator L defined by

$$Lu = -u'' ,$$

with boundary conditions

$$u(0) = u(a) = 0 .$$

Let K_0 be the integral operator

$$K_0 u = \int_0^a K_0(s, \theta) u(\theta) \, \mathrm{d}\theta ,$$

with

$$K_0(s, \theta) = \begin{cases} \dfrac{s}{a}(a - \theta) , & s \leq \theta \\ \dfrac{\theta}{a}(a - \theta) , & s > \theta . \end{cases}$$

Note that the kernel is symmetric so that the integral operator K_0 is self adjoint and hence its eigenvalues are real. One can verify readily that K_0 satisfies the equation

$$LK_0 u = u .$$

Let $w_n(s)$ denote the eigenfunctions of K_0 with corresponding eigenvalues α_n, viz.

$$K_0 w_n = \alpha_n w_n .$$

If we apply the operator L to the above equation we obtain

$$\frac{1}{\alpha_n} w_n = L w_n ,$$

so that the eigenfunctions w_n satisfy the differential equation

$$w_n'' + \frac{1}{\alpha_n} w_n = 0 ,$$

with boundary conditions

$$w_n(0) = w_n(a) = 0 .$$

We find immediately that α_n has to be of the form

$$\alpha_n = \frac{a^2}{(n\pi)^2} .$$

It follows that the eigenvalues of K_0 are strictly positive so that K_0 is positive definite.

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